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TRANSVERSE VIBRATIONS OF RECTANGULAR, TRAPEZOIDAL AND TRIANGULAR ORTHOTROPIC, CANTILEVER PLATES

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1. INTRODUCTION

Consider the cantilever plates depicted in Figure 1, in which rectangular, trapezoidal and triangular shapes are shown. It must be clarified that the analysis presented in this note is performed for isosceles triangular and trapezoidal plates, but that the general approach is certainly valid for more general shapes.

Rather extensive information on natural frequencies and mode shapes of these structural elements has been obtained by distinguished researchers in the case of isotropic plates, and has been compiled in Leissa's classical treatise [1]. On the other hand, and in view of the ever increasing use of non-isotropic materials in many technological applications, it is necessary to be able to predict, at least, the mechanical response of certain elements of these structures when subjected to static or dynamic loads.

This note presents a simple, general methodology for dealing with the orthotropic structural elements shown in Figure 1 when vibrating in their fundamental mode. Accordingly, the fundamental frequency coefficient is determined using the optimized Rayleigh–Ritz method [2] and trigonometric co-ordinate functions which contain, in their argument, optimization parameters that allow for minimization of the fundamental frequency once the determinantal equation is generated [3, 4]. A unique feature of the proposed technique is the fact that the dynamic problem for the three shapes shown in Figure 1 is accomplished in a unified fashion. Good agreement with predictions obtained by means of the finite element method is achieved [5].

2. APPROXIMATE ANALYTICAL SOLUTION

Consider first the cantilever, rectangular plate; see Figure 1(a). Following previous studies [3, 4], one approximates the fundamental mode shape by means of the expression

$$W(x, y) \simeq W_a(x, y) = \sum_{j=1}^{J} A_j \sin^2 \frac{\pi x}{\gamma_j a} \cos \frac{\pi y}{\gamma_{j+1} b}, \qquad \gamma_1, \gamma_2 > 1, \tag{1}$$

which satisfies the two essential boundary conditions at $x = 0$ and is symmetric with respect to the x-axis. The parameters γ_j and γ_{j+1} are, in this case, Rayleigh's optimization parameters, which allow for the minimization of the fundamental frequency.

Substituting equation (1) in the governing functional [6],

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$$J[W] = \tfrac{1}{2}\int_0^a \int_{-b/2}^{b/2} \left[D_1 \left(\frac{\partial^2 W}{\partial x^2}\right)^2 + 2D_1 \nu_2 \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + D_2 \left(\frac{\partial^2 W}{\partial y^2}\right)^2 \right.$$

$$\left. + 4D_k \left(\frac{\partial^2 W}{\partial x\, \partial y}\right)^2 \right] \mathrm{d}x\, \mathrm{d}y - \frac{\rho\omega^2}{2} h \int\int W^2\, \mathrm{d}x\, \mathrm{d}y, \qquad (2)$$

where $D_k = \frac{1}{2}(D_3 - \nu_2 D_1)$, and minimizing with respect to the A_j's, one generates a homogeneous, linear system of equations in the A_j's.

The non-triviality condition yields a secular determinant, the lowest root of which is the fundamental frequency coefficient $\Omega_1 = \sqrt{\rho h/D_1}\omega_1 b^2$.

Finally, one minimizes Ω_1 with respect to the γ's and an optimized value of the fundamental frequency coefficient is obtained.

Now consider the trapezoidal and triangular plates of Figure 1(b) and 1(c). It is certainly legitimate to use the functional relation (1) also as an approximation to the fundamental mode of vibration of these two types of non-rectangular shapes. Then, expression (1) is substituted in the energy functional (2) but, obviously, the integrations will be performed over the corresponding plate domains. The remainder of the computational procedure will be the same as in the case of the rectangular plate.

3. NUMERICAL RESULTS AND CONCLUSIONS

In Table 1 are depicted fundamental eigenvalues $\sqrt{\rho h/D_1}\omega_1 b^2$ in the case of (1) isotropic rectangular plates ($D_1 = D_2 = D_3 = D$; $\nu = 0{\cdot}30$) and (2) orthotropic plates ($D_2/D_1 = 1/2$;

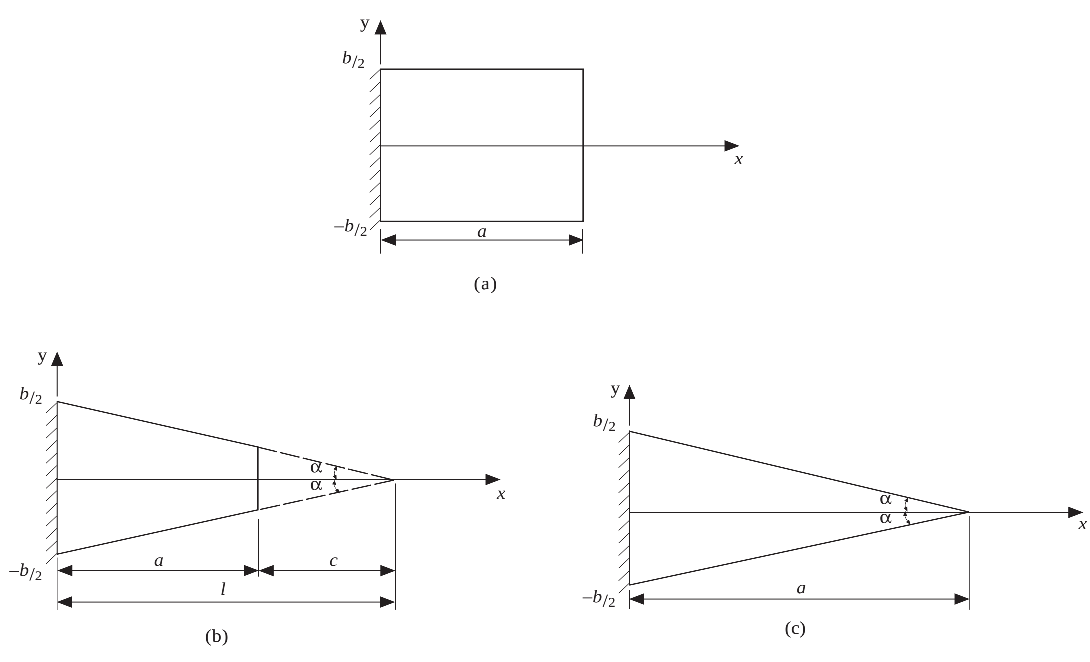

Figure 1. Cantilever orthotropic plates vibrating in their fundamental mode, considered in the present study. (a) Rectangular plate; (b) trapezoidal plate; (c) triangular plate.

Table 1

Fundamental frequency coefficients $\Omega_1 = \sqrt{\rho h/D_1}\omega_1 b^2$ *of cantilever, rectangular plates*

	Isotropic plate, $\nu = 0{\cdot}30$			Orthotropic plate $D_2/D_1 = 1/2$, $D_k/D_1 = 1/3$, $\nu_2 = 0{\cdot}30$	
a/b	[7]	Analytical approach	Finite element	Analytical approach	Finite element
2/5	21·9418	21·9552	21·8605 *(1392)	21·9514	21·7954 *(1392)
2/3	7·8804	7·8840	7·8419 *(1008)	7·8662	7·8014 *(1008)
1	3·4917	3·4959	3·4714 *(624)	3·4803	3·4432 *(624)
3/2	1·5454	1·5496	1·5349 *(1040)	1·5387	1·5160 *(1040)
5/2	0·5530	0·5558	0·5485 *(1456)	0·5497	0·5380 *(1456)

* Number of degrees of freedom.

$D_k/D_1 = 1/3$; $\nu_2 = 0{\cdot}30$). In the case of isotropic constitutive relations, present analytical results are in excellent agreement with those determined in reference [7] and the ones obtained in the present investigation using the finite element algorithm developed in reference [8]. The agreement is also very good in the case of orthotropic plates (the analytical results are compared with those obtained by means of the finite element code developed in reference [5] which is based on the very accurate isotropic element presented in reference [8]).

Table 2 deals with the isosceles trapezoidal shape; see Figure 1(b)*. First the isotropic plates ($\nu = 0{\cdot}3$) is considered and the values of $\sqrt{\rho h/D}\omega_1 l^2$ obtained by means of the proposed analytical approach are compared with those available in reference [1]. The agreement is, in general, very good. An exception is the configuration corresponding to $c/l = 0{\cdot}4$ since the proposed technique yields 24·706 and the value indicated in reference [1] is 18·397.

Presumably the latter is in error since for this configuration $a/b = 0{\cdot}40$, as indicated in Table 2. For this value of a/b and, since $l = b$, one has $\sqrt{\rho h/D}\omega_1 b^2 = 21{\cdot}9418$ for a rectangular plate [7]. Since a dynamic stiffening effect takes place for trapezoidal plate of the same value of a/b, it does not seem reasonable to have a value as low as 18·397 as indicated in reference [1]. Instead, it seems that 24·786 is a more rational figure (admittedly: this value is certainly an upper bound). Table 2 also contains fundamental eigenvalues in the case of orthotropic trapezoidal plates for different values of c/l and $\tan\alpha = 1/2$; see Figure 1(b). The rate of convergence between the one- and two-term analytical solutions seems reasonably good.

Finally, in Table 3 are presented values of $\sqrt{\rho h/D_1}\,\omega_1 a^2$ for triangular isotropic and orthotropic plates; see Figure 1(c).

In the case of isotropic plates some of the values obtained in the present investigation are lower than those available in reference [1]. Since the present approach yields upper bounds, one can conclude that in some instances the proposed technique yields results that are more accurate than those previously determined.

* Obviously the triangular, isosceles configuration constitutes a particular situation ($c/l = 0$).

TABLE 2
Fundamental frequency coefficients $\Omega_1 = \sqrt{\rho h/D_1}\omega_1 l^2$ *of trapezoidal plates* (*Figure* 1(*b*); $\tan\alpha = 1/2$)

	c/l	Reference [1]: Beam functions	Reference [1]: Polynomial approximation	Reference [1]: Lower bound	Present analytical approach: One term	Present analytical approach: Two terms
Isotropic plates, $\nu = 0{\cdot}30$	0·0	7·152 7·149	7·163	6·88	6·989	6·942
	0·2	8·465	8·15	8·042	8·238	8·165
	0·4	13·211	12·291	11·16	12·457	12·309
	0·6*	18·397	—	—	25·162	24·786
Orthotropic plates, $D_2/D_1 = 1/2$, $D_k/D_1 = 1/3$, $\nu_2 = 0{\cdot}30$	0·0				6·862	6·810
	0·2				8·105	8·024
	0·4				12·316	12·153
	0·6				25·029	24·635

* The value $c/l = 0{\cdot}6$ corresponds, for $\tan\alpha = 1/2$, to $a/b = 0{\cdot}40$ and $l = b$. For this value of the aspect ratio, the fundamental frequency coefficient of a rectangular plate is 21·941 [7]. Accordingly, for a trapezoidal plate of $a/b = 0{\cdot}40$ it does not appear reasonable that the value 18·397 from reference [1] is correct. Because of the dynamic stiffening effect, a value larger than 21·941 would be expected.

It is important to point out that if one expresses the fundamental frequency coefficient in terms of a, one has $\Omega_1 \simeq 3{\cdot}50$ for the rectangular plate and $\Omega_1 \simeq 7$ for the isosceles triangular plate. Accordingly, dynamic stiffening is achieved. The trapezoidal configuration will possess an intermediate value.

TABLE 3
Fundamental frequency coefficients $\Omega_1 = \sqrt{\rho h/D_1}\omega_1 a^2$ *of cantilever triangular, isosceles plates* (*Figure* 1(*c*))

		a/b = 1	2	4	7
Isotropic plate, $\nu = 0{\cdot}30$	Reference [1]	7·149	7·122	7·080	7·068
	Analytical approach				
	One term	6·989	7·108	7·132	7·137
	Two terms	6·942	7·063	7·083	7·084
Orthotropic plate, $D_2/D_1 = 1/2$, $D_k/D_1 = 1/3$, $\nu = 0{\cdot}30$	Analytical approach				
	One term	6·862	7·037	7·080	7·089
	Two terms	6·810	6·979	7·005	6·999

Implementation of the present technique has been greatly facilitated by the use of Mathematica™. Future studies will consider non-symmetrical structural elements.

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